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Simple dead-time corrections for discrete time series of non-Poisson data

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Abstract
The problem of dead time (instrumental insensitivity to detectable events due to electronic or mechanical reset time) is considered. Most existing algorithms to correct for event count errors due to dead time implicitly rely on Poisson counting statistics of the underlying phenomena. However, when the events to be measured are clustered in time, the Poisson statistics assumption results in underestimating both the true event count and any statistics associated with count variability; the 'busiest' part of the signal is partially missed. Using the formalism associated with the pair-correlation function, we develop first-order correction expressions for the general case of arbitrary counting statistics. The results are verified through simulation of a realistic clustering scenario.

Keywords: dead time, Poisson, discrete data

1. Introduction and motivation

The phenomena of 'dead time' (a time interval following a discrete detection event during which an instrument ceases to function) occurs in a wide variety of instruments (e.g., Mueller (1973)). Positron emission tomography (PET) detectors, atmospheric physics cloud probes and the common Geiger counter all are subject to dead time. While such instruments typically measure the rate of event counts (λ), the dead time causes some events to be missed and the rate *as detected* is lower than the true rate of the process.

Numerous studies over the last several decades addressed this underestimation of the true rate of the process, e.g., see Mueller (1973) for a classic paper on this subject or Brenguier (1989), and Brenguier and Amodai (1989) for more recent and specific treatment. The underlying assumption almost always made in these and similar studies, however, is that the system being examined obeys Poisson statistics (or, equivalently, the inter-event waiting time distribution is properly characterized by an exponential distribution).

However, there is often no *a priori* reason for making this assumption. In fact, there are common scenarios where we would expect the underlying time series—if not subject to dead time—would be expected to be more 'clustered' or 'clumpy' than a Poisson time series. If dead-time corrections to estimate the true event rate λ are implemented in these 'clumpy' time series, one will underestimate the true event rate λ and any bias any other statistics associated with λ.

This scenario is not of mere academic interest. Due to the 'quantum Zeno effect', deviations from the Rutherford decay law (and hence Poisson counting statistics) are expected (and observed) in nuclear and atomic decay measurements on some scales (see, e.g., Greenland (1988), Norman *et al* (1988), Concas and Lissia (1997), Curtis *et al* (1997), Facchi and Pascazio (1999), Fischer *et al* (2001)). Recent studies of time series of aerosols (e.g. Larsen *et al* (2003), Larsen (2007)), cloud droplets (e.g. Kostinski and Jameson (2000), Kostinski and Shaw (2001)) and rain drops (e.g. Kostinski and Jameson (1997), Kostinski *et al* (2006)) demonstrate identifiable non-exponential waiting times and are often measured with instruments that have dead-time effects.

Our goal here is to develop simple methods for retrieving the event rate λ and related second-order statistics for a time series recorded with an instrument subject to either one of the two most basic types of dead time. Our main tool is the pair-correlation function (PCF)—a function that directly quantifies departures from Poisson statistics as a function of scale. As far as we know, this is a novel application of the PCF formalism.

The next section develops the basic terminology associated with the different types of dead time and the mathematics associated with the pair-correlation function. We then proceed to establish, with the aid of PCF, closed-form estimates of the true count rate λ in terms of the measured count rate λ_m for a Poisson distribution. Next, simulations are presented for both Poisson and non-Poisson distributions subject to dead time of various types. This motivates the following section, where the poor estimate for the count rate based on Poisson statistics is improved by using some of the available information from the pair-correlation function. We conclude with comments regarding the applicability of these results.

2. Background

2.1. *Two types of dead time*

Using the basic outline developed by Brenguier and Amodei (1989), we will consider the two simplest varieties of dead time, which we call extensible and non-extensible. For an instrument with non-extensible dead time, there is a temporal period of fixed duration τ after an event detection during which the instrument is insensitive to subsequent events. In other words, given an event detection at time $t_\circ$ the instrument will be unable to detect events in the time interval $[t_\circ, t_\circ + \tau]$. This is called non-extensible dead time because the time period of insensitivity cannot be extended, even if an event occurs during the insensitive time interval. Note that many instruments also have coincidence errors that are usually treated separately. Many of these coincidence errors, however, can be considered mathematically equivalent to a short non-extensible dead time associated with the first of the two detections.

Conversely, an instrument with extensible dead time regains sensitivity (resets) only if no events (detected or not) have been encountered for a time interval of duration τ. (This can be conceived of 'transit time through an instrument' if it is detecting individual particles in time, 're-zeroing time' where an instrument requires a certain null signal to return before detecting the next event, or any number of other manifestations.) In principle, then, a sequence of closely spaced events can induce an insensitive time period exceeding τ. An illustration for both types of dead time is given in figure 1.

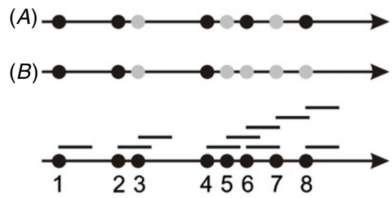

Figure 1. A cartoon to illustrate the conditions for retaining a detection due to an instrument with non-extensible (*A*) and extensible (*B*) dead times. The bottom row demonstrates the true distribution of event arrivals (time increasing to the right). The solid black dots indicate an event arrival, while the short horizontal lines above each solid black dot notate the time interval of length τ following the event. The distribution measured by the instrument with non-extensible dead time (*A*) misses the events marked 3 (the instrument remains insensitive due to detection of event 2), 5 (due to event 4) and 7 (due to event 6). Although event 6 follows event 5 by a time interval of less than τ, event 6 is detected since it follows the last *detected* event (4) by time interval exceeding τ. The instrument that has extensible dead time (*B*) misses all of the events following 4 due to the fact that there is no interval of length τ or longer where no events (retained or not) were present. (See the sequence of overlapping line segments above the initial distribution.) As such, the 'effective' dead time for the extensible dead time exceeds τ.

2.2. *The pair-correlation function*

To quantify the influence of the insensitive interval on the measured inter-arrival distribution function we shall use a pair-correlation function. Most simply, the pair-correlation function evaluated at some lag time t (written as $\eta(t)$) measures the probability 'enhancement' of an event being detected at $t_\circ + t$ given a detection at $t_\circ$. More explicitly,

$$p_t(t_\circ + t|t_\circ)\,\mathrm{d}t = \lambda(1 + \eta(t))\,\mathrm{d}t, \tag{1}$$

where $p_t(a|b)\mathrm{d}t$ is the probability of finding an event in a short time interval of duration $\mathrm{d}t$ starting at a given the detection of an event at time b. λ is the rate (number of events per unit time). For a perfectly random Poisson process, the probability of particle arrivals at $t_\circ$ and $t_\circ + t$ is independent; there is no lag-dependent enhancement and $\eta(t) \equiv 0$. As such, for a system exhibiting perfect randomness (one that follows a Poisson distribution with constant rate λ), we recover

$$p_t(t_\circ + t|t_\circ)\,\mathrm{d}t = \lambda\,\mathrm{d}t. \tag{2}$$

The function $\eta(t)$ can range from -1 (for exclusion, when an event at $t_\circ$ prevents an event anywhere in $[t_\circ + t, t_\circ + t + \mathrm{d}t]$ to $(\lambda \mathrm{d}t)^{-1} - 1$ (assuring the presence of an event somewhere in $[t_\circ + t, t_\circ + t + \mathrm{d}t]$, since this causes the rhs of equation (1) to return unity). Further background can be found in Cox and Isham (1980), Landau and Lifshitz (1980), Kostinski and Shaw (2001), Shaw *et al* (2002) or Larsen (2006).

There are several important reasons for introducing the pair-correlation function in our development here:

- The pair-correlation function is a memoryless scale-localized measure, identifying departures from perfect randomness as a function of scale.
- The pair-correlation function *directly* quantifies departures from perfect (Poisson) randomness.
- The pair-correlation function is linked in a known way to most other measures of temporal and spatial texture.
- The pair-correlation function is computationally easy to estimate.
- The pair-correlation function can be written directly in terms of waiting-time distributions.
- If the pair-correlation function is known or measured for a distribution, one can infer the variance to mean ratio as a function of scale by using the correlation-fluctuation theorem (see, e.g., Cox and Isham (1980), Landau and Lifshitz (1980)).
- When no event can possibly be detected in the interval $(t_\circ, t_\circ + \tau + \mathrm{d}t)$ due to instrument insensitivity, the pair-correlation function takes the known form $\eta(t < \tau) = -1$.

In practice, the pair-correlation function for a measured system can be estimated from the equation,

$$\eta(t) = \frac{d(t)}{r(t)} - 1, \quad (3)$$

where $d(t)$ is the total number of event pairs with separation between t and $t + \mathrm{d}t$, and $r(t)$ is the expected number of event pairs with separation between t and $t + \mathrm{d}t$ if one assumes the data at hand follows a homogeneous Poisson distribution (and use the measured rate λ and total sample duration T to infer $r(t)$).

Two of the other properties of the pair-correlation function mentioned above should be expanded upon. First, as noted by Picinbono and Bendjaballah (2005), the pair-correlation function can be written in terms of the waiting-time statistics for the distribution with

$$\eta(t) = -1 + \frac{1}{\lambda\,\mathrm{d}t} \sum_{k=1}^{\infty} f_k(t), \quad (4)$$

where $f_k(t)$ is the probability that the kth event posterior to a given event can be found in the interval $(t, t+\mathrm{d}t)$. To estimate the pair-correlation function, we can sum a finite number of terms of this expression. The accuracy of the approximation is then determined by the value of k used, the averaging scale $\mathrm{d}t$ and the scale of interest t. For some distributions at very small values of t, even the $k = 1$ term alone can be a suitable approximation.

The other important result requiring elaboration is the correlation-fluctuation theorem. As demonstrated by Cox and Isham (1980), Landau and Lifshitz (1980) and Larsen (2006), we can relate the variance/mean ratio as a function of scale to the integral of the pair-correlation function. In one dimension (suitable for time-series analysis), this relation reduces to

$$\frac{\mathrm{var}[N(t)]}{\overline{[N(t)]}} - 1 = \frac{2\lambda}{t} \int_0^t (t - t')\eta(t')\,\mathrm{d}t', \quad (5)$$

where $N(t)$ is the number of events occurring in a time interval of duration t, $\overline{[N(t)]}$ represents the average of this quantity (as a function of t) and $\mathrm{var}[N(t)]$ the associated variance of this quantity.

As expected, in the case of a Poisson process, $\eta(t) \equiv 0$ and $\mathrm{var}[N(t)] = \overline{[N(t)]}$ for all values of t. This is an important property as it allows computation of a bulk measurement relevant to statistical descriptions of spatio-temporal texture.

3. Corrections for the Poisson case

We shall now develop correction formulae for event count statistics based on the implicit assumption of a Poisson process. In particular, we seek relationships to infer λ (the true rate of events) in terms of λ_{m} (the rate as measured). Additionally, more complete information regarding the true statistical structure of the distribution can be inferred if some estimate of the pair-correlation function and variance/mean ratio of the initial distribution is obtained.

Formally, we construct a Poisson distribution as follows: let us first take a perfectly random sequence of events in the time interval $(0, T)$ with the events occurring at times $t_1, t_2, \ldots, t_N$ and let $\lambda = N/T$. 'Perfectly random' here is interpreted to mean that the event times are mutually independent and equally likely everywhere in the interval $(0, T)$. There is another sequence of *measured* events $t_{\mathrm{m}1}, t_{\mathrm{m}2}, \ldots, t_{\mathrm{m}M}$ with $\lambda_{\mathrm{m}} = M/T$ and $M \leqslant N$. Further, it is assumed that the instrument accurately records event times, but sometimes fails to record events due to the event occurring during a period of instrumental insensitivity. The characteristic time defining the dead time will be τ, and all variables associated with the measured distribution will be subscripted with m.

3.1. Non-extensible dead time

In the case of non-extensible dead time, an event is recorded at time $t_\circ$ if and only if (1) there is a member of the set t_i where $t_i = t_\circ$ and (2) no member of the set of measured events t_{m} is in the interval $[t_\circ - \tau, t_\circ)$.

It proves useful to evaluate the pair-correlation function for this system from the expression $\eta(t) = d(t)/r(t) - 1$. In general, for a Poisson process, the expected number of event pairs separated by times between $(t, t+\mathrm{d}t)$ is given by the total number of events multiplied by the probability that each event has a subsequent event arriving in the appropriate time interval after it—e.g. $r(t) = N \cdot \lambda\,\mathrm{d}t$. Similarly, $r_{\mathrm{m}}(t) = M \cdot \lambda_{\mathrm{m}}\,\mathrm{d}t$. Consequently, some means of finding λ_{m} in terms of λ is necessary.

In the case of non-extensible dead time, there is a total period of instrument insensitivity equal to $M\tau$. During that interval, we expect $\lambda \cdot M\tau$ events to have occurred and gone undetected. So out of a total of N events, M were kept, and the difference between N and M is given by $M\lambda\tau$. Recalling the total duration of the event is T, we know $N = \lambda T$ and $M = \lambda_{\mathrm{m}} T$ so we obtain the relationship

$$\lambda T - \lambda_{\mathrm{m}} T = \lambda\lambda_{\mathrm{m}} T\tau. \quad (6)$$

Solving for either λ or λ_{m} in terms of the other variables gives

$$\lambda_{\mathrm{m}} = \frac{\lambda}{1 + \lambda\tau}, \quad (7)$$

$$\lambda = \frac{\lambda_{\mathrm{m}}}{1 - \lambda_{\mathrm{m}}\tau}. \quad (8)$$

Since $r_{\mathrm{m}}(t) = M \cdot \lambda_{\mathrm{m}}\,\mathrm{d}t$ and $M = \lambda_{\mathrm{m}} T$, this allows a rewriting of $r_{\mathrm{m}}(t)$ without M as $r_{\mathrm{m}}(t) = \lambda_{\mathrm{m}}^2 T\,\mathrm{d}t$. Now only $d_{\mathrm{m}}(t)$ is yet required to find $\eta_{\mathrm{m}}(t)$ and, via use of the correlation-fluctuation theorem (equation (5)), also to find $[\mathrm{var}/\mathrm{mean}]_{\mathrm{m}}$ for the Poisson process measured by an instrument with non-extensible dead time.

By definition, it is known that $d_{\mathrm{m}}(t < \tau)$ is identically zero—if any two events are initially separated by an interval shorter than the dead time, at least one of the events will go undetected, leaving $\eta_{\mathrm{m}}(t < \tau) = -1$. The domain $t > \tau$ is more complicated. In the domain $\tau < t < 2\tau$, it can be argued that any event pair is retained if and only if the interval $(t_\circ + \tau, t_\circ + t)$ is devoid of any events. (Events in the interval $(t_\circ, t_\circ + \tau]$ may still be present and still allow retaining of an event in $(t_\circ + t, t_\circ + t + \mathrm{d}t)$ due to the non-extensible nature of this dead time.)

The probability that an event pair separated by $\tau < t < 2\tau$ is retained, then, can be computed by the probability that the event at $t_\circ$ is retained $(\lambda_\mathrm{m}/\lambda)$ multiplied by the probability that the event in $(t_\circ + t, t_\circ + t + \mathrm{d}t)$ is retained, given by the void probability in the interval $(t_\circ + \tau, t_\circ + t)$:

$$d_\mathrm{m}(\tau < t < 2\tau) = d(t)(\lambda_\mathrm{m}/\lambda)p_0(t-\tau) = d(t)(\lambda_m/\lambda)\exp(-\lambda(t-\tau)). \quad (9)$$

Since $d(t) = r(t)$ for a Poisson process, $d(t) = N\cdot\lambda\,\mathrm{d}t = \lambda^2 T\,\mathrm{d}t$. Using $d_\mathrm{m}(t)$ and $r_\mathrm{m}(t)$ to write $\eta_\mathrm{m}(t)$ via equation (3), we find

$$\eta_\mathrm{m}(\tau < t < 2\tau) = \frac{\lambda^2 T(\lambda_\mathrm{m}/\lambda)\exp(-\lambda(t-\tau))}{\lambda_\mathrm{m}^2 T} - 1 \quad (10)$$

and, using equation (8) to rewrite everything in terms of the measurable λ_m, this becomes

$$\eta_\mathrm{m}(\tau < t < 2\tau) = \left(\frac{1}{1-\lambda_\mathrm{m}\tau}\right)\exp\left(\frac{-\lambda_\mathrm{m}}{1-\lambda_\mathrm{m}\tau}(t-\tau)\right) - 1. \quad (11)$$

Closed-form expressions in the domain $t > 2\tau$ prove difficult to evaluate; one needs to compute the probability that given an event *measured* at $t_\circ$, there is an event present in $(t_\circ + t, t_\circ + t + \mathrm{d}t)$ *and* that it will be measured. Unlike the domain $(\tau < t < 2\tau)$, however, there is no simple way to assign this conditional probability. For completeness, a formula is presented in the appendix that uses the principle of induction to infer $\eta_\mathrm{m}(t > 2\tau)$. Practically speaking, empirical observations suggest that for an initially Poisson distribution, $\eta_\mathrm{m}(t > 2\tau)$ is not substantially different from zero. (This can be readily demonstrated with numerical simulation. It has also been physically observed in the raindrop arrival literature, e.g., Larsen *et al* (2005).) Thus, for the non-extensible initially Poisson case with $\lambda\tau \lesssim 0.4$, $\eta_\mathrm{m}(t)$ can be written to good approximation:

$$\eta_\mathrm{m}(t) = \begin{cases} -1 & \text{for} \quad t < \tau \\ \left(\frac{1}{1-\lambda_\mathrm{m}\tau}\right)\exp\left(\frac{-\lambda_\mathrm{m}}{1-\lambda_\mathrm{m}\tau}(t-\tau)\right) - 1 & \text{for} \quad \tau < t < 2\tau \\ 0 & \text{for} \quad t > 2\tau. \end{cases} \quad (12)$$

Using this approximate pair-correlation function to evaluate the variance/mean ratio from the correlation-fluctuation theorem in terms of observables yields (after laborious, but simple, application of equation (5) and rewriting in terms of observables)

$$\left[\frac{\mathrm{var}[N(t)]}{\overline{[N(t)]}}\right]_\mathrm{m} = 3 - 4\lambda_\mathrm{m}\tau - 2\exp\left(\frac{-\lambda_\mathrm{m}\tau}{1-\lambda_\mathrm{m}\tau}\right) + \frac{1}{t}\left[4\lambda_\mathrm{m}\tau^2 - \frac{2}{\lambda_\mathrm{m}} + \exp\left(\frac{-\lambda_\mathrm{m}\tau}{1-\lambda_\mathrm{m}\tau}\right)\left(2\tau + \frac{2}{\lambda_\mathrm{m}}\right)\right]. \quad (13)$$

For $t \gg \tau$ and $\lambda_\mathrm{m} t \gg 1$ only the first three terms are retained.

Although at this point the forward problem (finding the measured statistics from the initial properties of the distribution) has been solved, the real utility in making dead-time corrections is to try and infer the true population statistics from the measurements made by the instruments that are subject to dead-time criteria. Most importantly, we seek to infer the true concentration (related to λ) from the measured quantity λ_m. Luckily, for non-extensible dead time this relationship has been given by equation (8).

An accurate guess to the variance/mean ratio for the initial distribution would also be useful. A naive but (for a Poisson distribution) accurate correction is just to subtract off the known (incorrect) variance/mean ratio for the measured distribution and add the known true value of 1, e.g.,

$$\left[\frac{\mathrm{var}[N(t)]}{\overline{[N(t)]}}\right] \sim \left[\frac{\mathrm{var}[N(t)]}{\overline{[N(t)]}}\right]_\mathrm{m} - 3 + 4\lambda_\mathrm{m}\tau + 2\exp\left(\frac{-\lambda_\mathrm{m}\tau}{1-\lambda_\mathrm{m}\tau}\right) + 1. \quad (14)$$

Simulations in the following sections reveal that while this inversion works well for a Poisson process, it fails for systems that do not exhibit perfectly random behavior.

3.2. Extensible dead time

If the initial distribution of events is observed with a detector subject to extensible dead time, each event is detected if and only if there is an empty interval of minimal duration τ preceding it; $t_i \in \{t_\mathrm{m}\}$ iff $t_i - t_{i-1} \geqslant \tau$. Let the distribution function governing $t_i - t_{i-1}$ be written as $f_1(t)\,\mathrm{d}t$, which—for the Poisson distribution describing a perfectly random sequence of events—takes the form

$$f_1(t)\,\mathrm{d}t = \lambda\exp(-\lambda t)\,\mathrm{d}t. \quad (15)$$

This definition can be used to determine the pair-correlation function for the measured distribution. For $t < \tau$, there are no retained event pairs with separation t and, hence, $d_\mathrm{m}(t < \tau) = 0$ and $\eta_\mathrm{m}(t < \tau) = -1$ (from equation (3)).

For $t > \tau$, the knowledge that there is an event at $t_\circ$ has (by construction) no bearing on whether an event at $t_\circ + t$ is preceded by an empty interval of duration τ. Consequently, the number of retained event pairs separated by t can be found by taking the initial number of event pairs separated by t and multiply by the (independent) probabilities that each of the two events were retained. The probability that *any* event is retained can be evaluated from $\int_\tau^\infty f_1(t)\,\mathrm{d}t = \exp(-\lambda\tau)$. This leaves $d_\mathrm{m}(t) = d(t)\exp(-\lambda\tau)\exp(-\lambda\tau) = d(t)\exp(-2\lambda\tau)$. Finding $\eta_\mathrm{m}(t)$ now merely requires computation of $r_\mathrm{m}(t > \tau)$—the expected number of events separated by a time lag in the range of $(t, t+\mathrm{d}t)$ in a Poisson distribution of duration T and intensity λ_m. This can be computed from

$$r(t) = N\lambda\,\mathrm{d}t = \lambda^2 T\,\mathrm{d}t, \quad (16)$$

since there are a total of $N = \lambda T$ events, and the probability of finding a match for each one is given by $\lambda\,\mathrm{d}t$ for a Poisson distribution. Similarly, the measured distribution has Poisson expectation $r_\mathrm{m}(t) = M\lambda_\mathrm{m} = \lambda_\mathrm{m}^2 T$. This implies that $r_\mathrm{m}(t) = r(t)\cdot(\lambda_\mathrm{m}^2/\lambda^2)$. However, since the probability that each event pair is retained is given by $\int_\tau^\infty f_1(t)\,\mathrm{d}t = \exp(-\lambda\tau)$, it is known that $\lambda_\mathrm{m}/\lambda = \exp(-\lambda\tau)$ and $r_\mathrm{m}(t) = \exp(-2\lambda\tau)r(t)$.

Thus, in the range $t > \tau$ for a Poisson distribution examined with extensible dead time:

$$\eta_m(t > \tau) = \frac{d_m(t > \tau)}{r_m(t > \tau)} - 1 = \frac{d(t)\exp(-2\lambda\tau)}{r(t)\exp(-2\lambda\tau)} - 1 = 0. \tag{17}$$

In the initial distribution, $\eta(t) \equiv 0 \forall t$ implying that $d(t) = r(t)$. This gives the (exact) pair-correlation function:

$$\eta_m(t) = \begin{cases} -1 & \text{for} \quad t < \tau \\ 0 & \text{for} \quad t > \tau. \end{cases} \tag{18}$$

This matches the form hypothesized by Kostinski and Shaw (2001) when trying to account for the effect of instrumental dead time in clouds. The correlation-fluctuation theorem for $t > \tau$ then gives

$$\left[\frac{\text{var}[N(t)]}{\overline{[N(t)]}}\right]_m = 1 - 2\lambda_m\tau + \frac{\lambda_m\tau^2}{t}, \tag{19}$$

and for $t \gg \tau$ only the first two terms are retained.

Once again, inversion formulae are the true goal of this analysis. It was already calculated that the probability any given event is retained in the presence of extensible dead time can be computed from $\int_\tau^\infty f_1(t)\,dt = \exp(-\lambda\tau)$, implying $\lambda_m = \lambda\exp(-\lambda\tau)$. Exact inversion of this formula is not possible in closed form. An algebraic inversion formula (similar to that for non-extensible dead time) would be useful.

One possible approximate solution is to posit the ansatz:

$$\lambda \approx \frac{\lambda_m}{1 - \lambda_m\tau_e}, \tag{20}$$

where $\tau_e > \tau$ takes on the role of an 'effective' dead time accounting for the fact that the true dead time in the extensible case can be longer than τ if several events are close together (see figure 1). Plugging this ansatz in for λ in the exact formula $\lambda_m = \lambda\exp(-\lambda\tau)$ and solving for τ_e, we find

$$\lambda_m = \frac{\lambda_m}{1 - \lambda_m\tau_e}\exp\left(\frac{-\lambda_m\tau}{1 - \lambda_m\tau_e}\right). \tag{21}$$

Taking the first-order expansion (assuming $\lambda_m\tau$ is small) and solving the quadratic equation reveals the following approximation for τ_e:

$$\tau_e = \frac{1 \pm \sqrt{1 - 4\lambda_m\tau}}{2\lambda_m}. \tag{22}$$

Given that $\tau_e = 0$ when $\tau = 0$, this implies that the solution with the minus sign must be retained and the final approximation results in

$$\lambda \sim \frac{2\lambda_m}{1 + \sqrt{1 - 4\lambda_m\tau}}. \tag{23}$$

(Using the positive square root results in $\tau_e \sim (\lambda_m)^{-1} - \tau$ and, in turn, $\lambda \sim \tau^{-1}$ suggesting that the dead time is removing a very large fraction of the data. Although dead-time errors may not always be negligible, most measurements are able to avoid this problem and an in depth investigation of this solution is beyond the scope of this paper.)

As in the non-extensible case, a first-order correction for the variance/mean ratio can be easily inferred:

$$\left[\frac{\text{var}[N(t)]}{\overline{[N(t)]}}\right] \sim \left[\frac{\text{var}[N(t)]}{\overline{[N(t)]}}\right]_m + 2\lambda_m\tau. \tag{24}$$

Table 1. A comparison between the true and estimated values of λ and the variance/mean ratio using the inversion formulae developed in the text.

Dead time applied	λ_{inv}	$\left[\frac{\text{var}}{\text{mean}}\right]_{inv}$
None	1.0000	1.0000
Non-extensible	1.0002	1.0123
Extensible	1.0155	1.0121

4. Numerical simulations

The inversion expressions in the previous section were developed to demonstrate that results similar to 'traditional' inversions can also be developed via the pair-correlation function. The power of the PCF approach will be evident when we deal with the non-Poisson data. For now, to test the inversions already at hand, we carry out a numerical study with simulated data to demonstrate that, in fact, the underlying properties of a Poisson distribution can be inverted with dead-time 'tainted' data. Then we simulate a non-Poisson data-set and demonstrate how flawed these inversions can be.

4.1. Poisson simulation

A Poisson distribution with $\lambda = 1$ and $T = 1 \times 10^7$ was generated. (The times have all been non-dimensionalized and scaled to the mean rate) From this data, the pair correlation was estimated (using equation (4) with $k = 30$) for scales up to 1.5 times the average inter-event spacing. The variance to mean ratio as a function of scale for time scales ranging from $100/\lambda$ to $10000/\lambda$ was also computed. As expected, the pair-correlation function was effectively 0 for all times and the var/mean ratio very nearly unity for the entire range of scales.

The initial system was then 'measured' with simulated instruments subject to the two different types of dead time. In both cases, the dead time τ was set to 0.15. These numerical instruments obeyed the formal conditions outlined earlier (for example, an event is kept in the extensible case if and only if no other even preceded it by an interval of τ or less). Then λ_m, $\eta_m(t)$ and $(\text{var/mean})_m$ were computed for the post-probed data-sets. Results can be seen in figures 2 and 3.

Inserting $\tau = 0.15$ and $\lambda = 1$ into the theoretical expressions developed in the previous section results in the *a posteriori* estimates of λ (labeled λ_{inv}) and var/mean (labeled $[\text{var/mean}]_{inv}$) in table 1.

All of the measured quantities (concentration, pair-correlation function and variance/mean ratios) give excellent agreement when compared with the theoretical expressions developed in the previous sections. If this were the only simulation attempted, one might erroneously infer that these simple corrections (and the similar results developed elsewhere) do an excellent job of accounting for instrumental dead time.

4.2. Non-Poisson simulation

In order to examine how dead time influence a distribution that does not exhibit perfect randomness, a one-dimensional Matérn cluster process was also simulated. Although typically

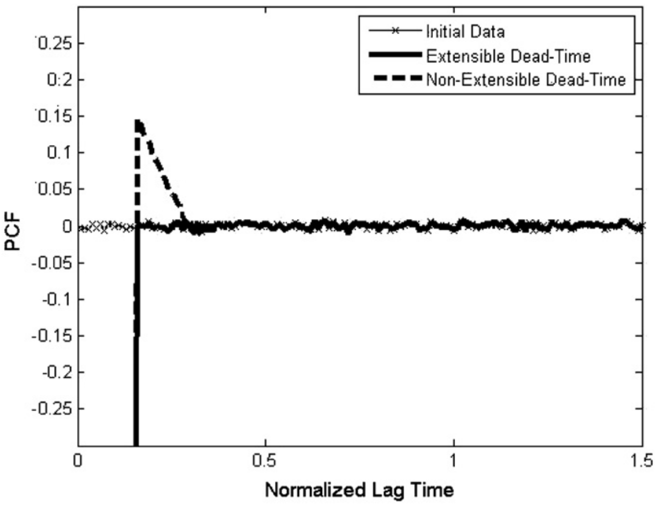

Figure 2. The pair correlation shown as a function of separation time for a simulated Poisson distribution. The *x*-axis is normalized so that the average inter-event time in the initial distribution is unity. The initial Poisson distribution was then 'measured' with simulated detectors with extensible and non-extensible dead times. The pair-correlation function of the data-sets as observed by the probes are also displayed. The dead-time parameter τ for both of the probes in this system has $\tau = 0.15$. Note that the pair-correlation function behaves as expected in the text; $\eta_{\text{initial}}(\forall t) = 0$, $\eta_{\text{extensible}}(t < \tau) = -1$, $\eta_{\text{extensible}}(t > \tau) = 0$, $\eta_{\text{non-extensible}}(t < \tau) = -1$ and $\eta_{\text{non-extensible}}(t > \tau)$ exhibits behavior consistent with our expectation, including $\eta_{\text{non-extensible}}(t = \tau) \sim (\lambda/\lambda_m) - 1 \sim 0.15$. See the text for details regarding how these results are obtained from theoretical considerations.

used for two- or three-dimensional systems, the Matérn cluster process is used here because it has a known analytical form of the pair-correlation function (see, e.g., Stoyan and Stoyan (1994), Stoyan *et al* (1987)) as well as possibly giving a physical description to some breakdown processes in nuclear science and atmospheric physics (see, e.g., Larsen (2006)).

This distribution is constructed by first generating a Poisson distribution with some fixed rate λ/λ_e. Each member of this distribution is considered a 'parent'. For each of the $T\lambda/\lambda_e$ parents, there is a Poisson-distributed number of 'daughters' (varying from parent to parent but drawn from the same distribution with mean λ_e) placed with uniform probability in the interval $(t_i - R, t_i + R)$ where the parent's arrival time is specified by t_i with $i \in [1, T\lambda/\lambda_e]$. The parents are then removed from the resulting distribution and the set of daughters mark the time of the events in the simulation.

The pair-correlation function for this distribution can be written in terms of R and λ_e (the mean number of daughters per parent) *only*. Using the parameters set in the simulation, the pair-correlation function follows

$$\eta(t) = \begin{cases} 2(1-t) & \text{for} \quad t < 1 \\ 0 & \text{for} \quad t > 1. \end{cases} \tag{25}$$

For $t \gg 1$ this leaves the variance to mean ratio as

$$\frac{\text{var}}{\text{mean}} = 1 + \frac{2\lambda}{t}\int_0^t \eta(t')(t-t')\,\mathrm{d}t' = 1 + 2\lambda - \frac{2\lambda}{3t} \sim 3. \tag{26}$$

Like the Poisson simulation, this distribution was generated so that the mean inter-event time was normalized to unity. However, due to computational concerns, only approximately 1 million of the daughter particles were in the final distribution (instead of the 10 million particles used in the Poisson simulation).

As was done in the Poisson simulation, the pair-correlation function was computed (using equation (4) with $k = 30$) for scales up to 1.5 times the average inter-event time. Then, the variance to mean ratio as a function of scale for times ranging from $100/\lambda$ to $10000/\lambda$ was computed. As expected, both the pair-correlation function and the variance/mean ratios match the theoretical form for the distribution prior to probing—as can be verified by examining figures 4 and 5.

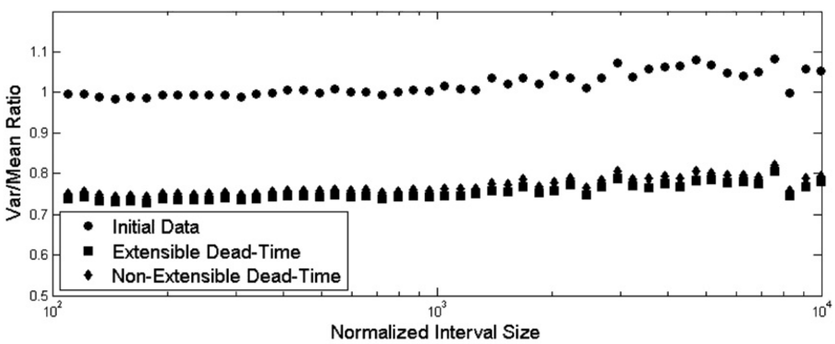

Figure 3. The variance/mean ratio plotted as a function of scale for the simulated distributions in figure 2. As expected, the ratio remains approximately unity for the true distribution. The correlation-fluctuation theorem predicts that this var/mean ratio for non-extensible dead time should be $3 - 4\lambda_m\tau - 2\exp(-\lambda_m\tau/(1-\lambda_m\tau)) \sim 0.76$ and the ratio for extensible dead time should be $1 - 2\lambda_m\tau \sim 0.74$ which again matches well with our reported results here. The slight difference between the two cases can be attributed to the slightly longer effective dead time and the slightly lower value of λ_m in the extensible case. Note that, while the var/mean ratio should remain constant for all $t \gg \tau$ for all of these curves, there is a slight amount of scatter for long times as there are fewer intervals of duration t when t becomes comparable to T, allowing for sampling fluctuations (i.e. shot noise).

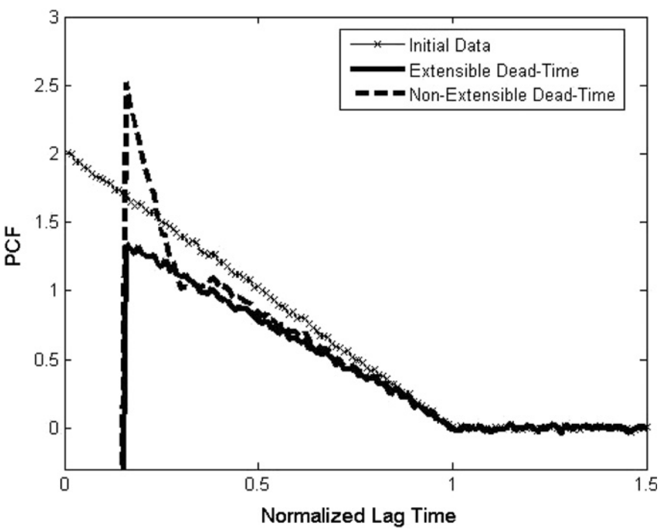

Figure 4. The pair-correlation function shown as a function of separation time for a simulated one-dimensional Matérn process. As in the Poisson case, the x-axis is normalized so that the average inter-event time in the initial distribution is unity. The generated distribution was then 'measured' with simulated probes having extensible and non-extensible dead times, respectively. The pair-correlation functions of the resulting distributions are displayed. Note that, in addition to the expected hump in the non-extensible dead-time case for $t \gtrsim \tau$, the pair-correlation function is now altered at scales substantially longer than the dead-time scale of $\tau = 0.15$ used. In particular, the simple assumption of $\eta_m(t < \tau) = -1$, $\eta_m(t > \tau) \sim \eta(t > \tau)$ implicitly made in Kostinski and Shaw (2001) is false. An important conclusion is that the presence of dead time influences the statistics of the measured distributions on scales substantially longer than τ.

The algorithm that simulates the measurement process with instruments subject to both extensible and non-extensible dead time was then used. Unlike the Poisson case, *a priori* theoretical predictions regarding how the pair-correlation function and the var/mean ratio should change from the probing process do not exist. Existing correction techniques in the literature implicitly assume either (1) such a clustered distribution is unphysical—all event distributions are inherently Poisson or that (2) the Poisson corrections developed in the earlier sections of this paper (and the associated equivalent corrections developed elsewhere) give a suitable first-order correction for inversions to determine λ and [var/mean] for any *realistic* distributions. This second point is a difficult argument to make since most measurements of the true distributions are made with instruments subject to dead time. Consequently, any blanket assumption about how much the dead-time effect influence the real physical system is untestable. It may only be with the advent of instruments using a fundamentally different measurement technique (e.g., Fugal *et al* (2004) for cloud physics) that a definitive answer to 'how substantial are the natural deviations from perfect randomness?' can be made.

One study that tried to investigate the possible effects of probing a clustered distribution (Kostinski and Shaw 2001) made the naive guess that the probing process changes the pair-correlation function in the following way:

$$\eta_m(t) = \begin{cases} -1 & \text{for} \quad t < \tau \\ \eta(t) & \text{for} \quad t > \tau \end{cases} \tag{27}$$

(e.g. it was assumed that the pair-correlation function accounts for mutual exclusion within the dead time but remains unchanged for timescales longer than τ). The figures presented here for the Matérn cluster process suggest that this ansatz is incorrect for both extensible and non-extensible dead-time forms. In the case of extensible dead time for $t > \tau$ the measured pair-correlation function is repressed below the true distributional value. Similarly, though the pair-correlation function at $t = \tau + \epsilon$ (ϵ an infinitesimal time increment) for the non-extensible cases exceeds the true distributional value (as might be expected from the results of the Poisson case), the pair-correlation function for $t \gtrsim 1.5\tau$ is underestimated.

Given the repression due to the measurement process, it comes as little surprise that the variance/mean ratios in figure 5 are also underestimated for the measured distributions.

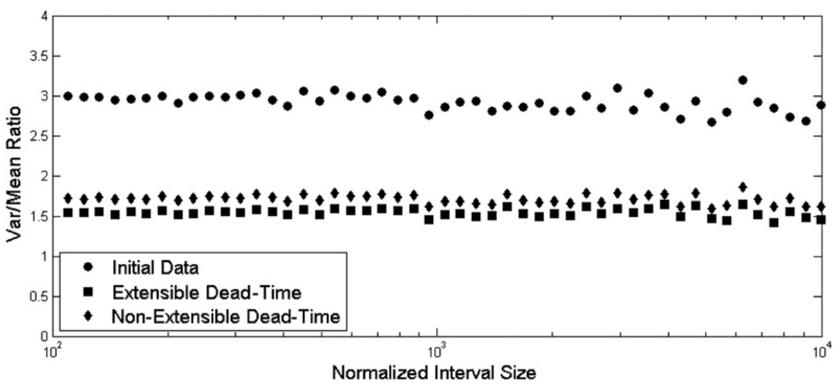

Figure 5. The variance/mean ratio plotted as a function of scale for the distributions associated with the Matérn simulation introduced in figure 4. Integration of the closed-form pair-correlation function gives a theoretical value of var/mean $= 2\lambda + 1 = 3$. Note that λ_m for the extensible dead-time case is found to be about 0.67 empirically; though only 1/3 of the events are removed, the large-scale variance/mean ratio is reduced from a substantially non-Poisson value around 3 to approximately half of the true value. If we erroneously assumed that the dead-time effect does not influence the statistics for scales longer than $t = \tau$, then we would have assumed $\eta_m(t < \tau) = -1$, $\eta_m(t > \tau) \sim \eta(t > \tau)$ and found the variance/mean ratio to be approximately 2.15 instead of 3.

Table 2. Attempts at inversion for the Matérn cluster process. An ideal inversion would have $\lambda = 1$ and var/mean = 3. The inversions without an asterisk correspond to the values obtained when using the Poisson inversion algorithm developed in section 3, while the asterisked columns correspond to the modified inversions as described in section 5.

Dead time applied	λ_{inv}	$(\lambda_{inv})^*$	$[\mathrm{var/mean}]_{inv}$	$[\mathrm{var/mean}]^*_{inv}$
None	0.9974	0.9974	2.9197	2.9197
Non-extensible	0.7956	1.0089	1.9112	2.8466
Extensible	0.7497	0.9029	1.7432	2.7568

Using the Poisson formula to correct for dead time in non-Poisson distributions results in faulty estimation of the true concentration (λ), pair-correlation function ($\eta(t)$) and the variance/mean ratio! This is established explicitly in table 2, where the inversions implied by equations (8), (14), (23) and (24) are used to tabulate λ_{inv} and $[\mathrm{var/mean}]_{inv}$.

5. Dead-time corrections for the non-Poisson case

It is apparent from table 2 that the inversion formulae developed for the case of a Poisson distribution subject to dead-time laced measurement are not suitable for distributions that are not Poisson. A more general inversion must be developed. An ideal inversion formula would give better estimation of the non-Poisson cases while (1) not altering the inversion for the Poisson cases and (2) remaining as assumption free as possible.

Logically, it is not hard to understand why the tabulated estimation of the true concentration (hereafter λ_{inv}) is underestimated for clustered distributions when using the Poisson inversion formulae. The nature of a Poisson distribution implied that the probability of an event being in the 'dead' measurement interval is neither more nor less likely than the total 'dead' time divided by the total sample duration T. For the Matérn simulation—as well as most other realistic non-Poisson statistical structures—we see that events tend to cluster. The observation of a single event not only triggers a period of instrumental insensitivity, but also indicates that subsequent event arrivals are more probable than perfectly random Poisson statistics imply. *We are missing the 'busiest' parts of the signal.* Mathematically, this manifests itself through a positive pair-correlation function in the dead interval ($\eta(t < \tau) > 0$).

Empirical observations (see, e.g., Shaw *et al* (2002), Larsen *et al* (2003, 2005)) suggest that realistic pair-correlation functions often are monotonically decreasing with scale, much like the Matérn cluster process used in the simulation here. Additionally, theoretical and computational studies (e.g., Reade and Collins (2000), Balkovsky *et al* (2001), Chun *et al* (2005)) suggest that under many circumstances the radial distribution function (a spatial analog of $\eta(t) + 1$) follows a decaying power-law behavior for inertial particles suspended in a turbulent fluid (one application of these types of detectors). Given a measured pair-correlation function that resembles either the solid or dotted lines in figure 4, it would be reasonable to suspect that the general trend observed for scales larger than $t = \tau$ (except for the peak at τ observed in the non-extensible case) likely would extend to smaller scales had the dead time not removed that part of the distribution.

Similarly, we would expect the magnitude of the true pair-correlation function to be higher than that observed in the sample distribution. Note that $\lambda > \lambda_m$ implies $r(t) > r_m(t)$, so the denominator in equation (3) is artificially repressed in the measured distribution. These two factors suggest a way to get a higher lower bound for the inverted concentration and variance/mean ratios than currently obtained using the Poisson inversion technique.

5.1. Extensible dead time

The previous estimate for λ_{inv} in the extensible case implicitly assumes that, though $\eta_m(t < \tau) = -1, \eta(t < \tau) = 0$. A better, yet still incorrect, estimate is that $\eta(t < \tau) = \eta_m(\tau + \epsilon)$ (ϵ a small positive constant to avoid the discontinuity in $\eta(t)$ at τ; hereafter, the addition of ϵ to τ when evaluating η is implied). This correction for $t < \tau$ still underestimates the pair-correlation function for all scales less than τ if η is monotonic, but it does partially account for the fact that the excluded region is more likely to have (undetected) events than in the Poisson case.

To help modify the inversion, make the approximation:

$$\eta(t) \approx \begin{cases} \eta_m(\tau) & \text{for} \quad t < \tau \\ \eta_m(t) & \text{for} \quad t > \tau. \end{cases} \tag{28}$$

This is still an imperfect estimate, but should increase the accuracy of the inversions λ_{inv} and $[\mathrm{var/mean}]_{inv}$. Estimating how many events are removed from each detected event in the extensible case amounts to computing:

$$\int_0^{\tau_e} \lambda[1 + \eta(t')]\,\mathrm{d}t' \sim \lambda\tau_e[1 + \eta_m(\tau)]. \tag{29}$$

So, in total time T, there are $\lambda_m T$ measured events and approximately $\lambda_m T \lambda \tau_e[1 + \eta_m(\tau)]$ undetected events. The number of detected plus undetected events should equal the total number of events λT; thus,

$$\lambda_m T[1 + \lambda\tau_e(1 + \eta_m(\tau))] = \lambda T, \tag{30}$$

$$(\lambda_{inv})^* \sim \frac{\lambda_m}{1 - \lambda_m\tau_e(1 + \eta_m(\tau))}, \tag{31}$$

where $(\lambda_{inv})^*$ denotes the modified inversion taking a better lower bound for $\eta_m(t < \tau)$ into account. τ_e is assumed the same as that obtained in doing the Poisson inversion. (Practically, $\tau_e \sim \tau$ so a correction on this would be a second-order effect) As can be seen in table 2, this new inversion formula substantially increases the accuracy of the inversion over λ_{inv}.

5.2. Non-extensible dead time

Once again, an instrument with non-extensible dead time is slightly more complicated than the extensible case due to the behavior of $\eta_m(\tau < t < 2\tau)$. Using $\eta(2\tau)$ may be a valid option, but given the rapid decay of the pair-correlation function observed in many physical systems, this may not give a particularly useful lower bound for the real pair-correlation function.

For lack of a better option, one possibility is to use $\eta_m(t < \tau) = 0.5(\eta_m(\tau) + \eta_m(2\tau))$. This mitigates the underestimation inevitable in using $\eta_m(2\tau)$ by simultaneously using the overestimate $\eta_m(\tau)$. This is somewhat arbitrary, but ultimately *no more arbitrary than assuming $\eta(t < \tau) = 0$ in a system known or suspected to be non-Poisson.* Similar calculations as those utilized in the extensible case yield

$$(\lambda_{inv})^* \sim \frac{\lambda_m}{1 - \lambda_m\tau(1 + 0.5(\eta_m(\tau) + \eta_m(2\tau)))}, \tag{32}$$

which, as in the extensible case, results in a much improved estimate for the true concentration of the initial distribution.

5.3. Inversion of second-order statistics: var/mean ratios

The correlation-fluctuation theorem states that

$$\left[\frac{\text{var}}{\text{mean}}\right]_m = 1 + \frac{2\lambda_m}{t}\int_0^t (t - t')\eta_m(t')\,dt', \tag{33}$$

where $\eta_m(t)$ is the measured pair-correlation function. If we note that (for both types of dead time) $\eta_m(t < \tau) = -1$ we can further write (eliminating terms of order τ/t)

$$\left[\frac{\text{var}}{\text{mean}}\right]_m = 1 - 2\lambda_m\tau + \frac{2\lambda_m}{t}\int_\tau^t \eta_m(t')(t - t')\,dt'. \tag{34}$$

A better overall estimate of the variance/mean ratio, however, could be brought about by writing:

$$\left[\frac{\text{var}}{\text{mean}}\right]_m^* = 1 + \frac{2(\lambda_{inv})^*}{t}\int_0^t (t - t')\eta_m(t')\,dt' \tag{35}$$

with $\eta_m(t')$ taking the modified forms in the above sections. Completing this computation and writing the new var/mean ratio in terms of the measured var/mean ratio, we obtain

$$\left[\frac{\text{var}}{\text{mean}}\right]_{inv}^* = \begin{cases} \frac{(\lambda_{inv})^*}{\lambda_m}\left\{\left[\frac{\text{var}}{\text{mean}}\right]_m - 1 + 2\lambda_m\tau\right\} \\ \quad + 1 + (\lambda_{inv}^*)\tau(\eta_m(\tau) + \eta_m(2\tau)) \quad \text{non-extensible} \\ \frac{(\lambda_{inv})^*}{\lambda_m}\left\{\left[\frac{\text{var}}{\text{mean}}\right]_m - 1 + 2\lambda_m\tau_e\right\} \\ \quad + 1 + 2(\lambda_{inv})^*\tau_e\eta_m(\tau) \quad \text{extensible.} \end{cases} \tag{36}$$

These new formulae give the estimates for $(\lambda_{inv})^*$ and $[\text{var/mean}]_{inv}^*$ reported in table 2. Note the substantial increase in accuracy over the inversions using the Poisson assumption (given in the columns marked λ_{inv} and $[\text{var/mean}]_{inv}$).

6. Concluding remarks

We have demonstrated definitively that using a Poisson-based inversion for the event-rate λ can substantially underestimate the true event rate for a non-Poisson data-set. Further, using the pair-correlation function formalism and some general trends regarding the typical shape of most physical data, we were able to improve the inversion offered to us by the traditional inversion formulae. Although the resulting closed-form expressions are approximate and not particularly elegant, they are reasonably simple to use and can substantially improve the accuracy of an event rate retrieval, as our simulations demonstrate.

The pair-correlation function used in the simulation of a non-Poisson system was representative for some physical systems in the atmospheric sciences (see, e.g., Larsen (2006)). Other similar pair-correlation functions (often taking a power-law form) are used in several of the treatments to describe the quantum Zeno effect in atomic and nuclear systems (see, e.g., Arbo *et al* (2000) and Garcia-Calderon *et al* (2001)).

Finally, there are several physical systems (e.g. positron emission tomography) where the legitimacy of the Poisson correction for dead time is still questioned. By using the pair-correlation function motivated correction formulae developed here, perhaps we can help bound the counting errors from those instruments.

Acknowledgments

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Appendix

In the main text, it was shown that the pair-correlation function for a Poisson process measured with an instrument subject to non-extensible dead time was determined in the range $0 < t < 2\tau$:

$$\eta_m(t) = \begin{cases} -1 & \text{for} \quad t < \tau \\ \left(\frac{\lambda}{\lambda_m}\right)\exp(-\lambda(t - \tau)) - 1 & \text{for} \quad \tau < t < 2\tau. \end{cases} \tag{A.1}$$

The principle of mathematical induction proves useful to find the pair-correlation function for lag times longer than 2τ. The probability of finding an event in $(t_\circ + t, t_\circ + t + dt)$ in the measured distribution, given an event at $t_\circ$, can be written:

$$p_m(t_\circ + t|t_\circ)\,dt = \lambda_m(1 + \eta_m(t))\,dt. \tag{A.2}$$

In words, $p_m(t_\circ + t|t_\circ)\,dt$ can also be interpreted as the probability that there is an event in $(t_\circ + t, t_\circ + t + dt)$ multiplied by the probability that there are no events recorded in $(t_\circ + t - \tau, t_\circ + t)$.

The probability of finding an event within $(t_\circ + t, t_\circ + t + dt)$ is, for a Poisson distribution, equal to $\lambda\,dt$ for small dt. (The use of a Poisson distribution here is limiting, but the best we can do in an assumption-free way and still should be good for a first-order correction) The probability that there are no measured events in $(t_\circ + t - \tau, t_\circ + t)$ is equal to the joint probability that each sub-interval of size dt does not have a measured event; independent joint probabilities imply multiplication and thus

$$p_m(t_\circ + t|t_\circ)\,dt = (\lambda\,dt) \cdot \left[\lim_{dt\to 0} \prod_{k=1}^{\tau/dt} (1 - \lambda_m\,dt\,[1 + \eta_m\,(t - (k-1)\,dt)]) \right]. \quad (A.3)$$

Combining equations (A.2) and (A.3) and solving for $\eta_m(t)$,

$$\eta_m(t) = -1 + \frac{\lambda}{\lambda_m} \lim_{dt\to 0} \times \left\{ \prod_{k=1}^{\tau/dt} [1 - \lambda_m\,dt\,(1 + \eta_m\,(t - (k-1)\,dt))] \right\}. \quad (A.4)$$

For a Poisson process, expressions for $\eta_m(\tau < t < 2\tau)$ and λ in terms of λ_m were derived in the main text. Numerical solution of the above equation should yield a reasonable estimate for $\eta_m(t > 2\tau)$. For a non-Poisson process, a numerical estimate of $\eta_m(t)$ can be made if some other means can be used to determine λ and $\eta_m(0 < t < \tau)$.

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